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The structure of n-point functions of chiral primary operators in $\mathcal{N} = 4$ super Yang-Mills at one-loop

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Abstract: We develop a compact representation of the one-loop n-point functions of all chiral primary operators in planar SU(N), $\mathcal{N} = 4$ super Yang-Mills theory in terms of tree-level disk correlation functions and the scalar one-loop box integral. As a check, known results for all four-point functions and for n-point extremal and near-extremal correlators are rederived. The result is then used to evaluate explicitly a selection of five and six-point functions. Our findings suggest that a general one-loop five-point function may be represented through the minimal four-point and five-point functions of weight two operators.

Keywords: AdS-CFT Correspondence, 1/N Expansion, Supersymmetric gauge theory

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1 Introduction and conclusions

The string-gauge theory duality in its best understood and most symmetric formulation identifies $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with type IIB superstrings on an $AdS_5 \times S^5$ background [1]. Great progress in our understanding of this AdS/CFT system has been achieved in recent years by exploiting its powerful superconformal symmetry together with the discovered integrable [2] structures. From the gauge theory perspective the set of local observables is given by n-point functions of gauge invariant composite operators which fall into multiplets of the superconformal symmetry group. In the early days of the AdS/CFT correspondence the study of so called chiral primary operators, which are scalar composite operators in the symmetric traceless representation of the SU(4) R-symmetry group with Dynkin labels $[0, k, 0]$, received great attention. These BPS operators are annihilated by one-half of the Poincare supercharges and are dual to the infinite tower of

Kaluza-Klein modes of the $AdS_5 \times S^5$ compactification of IIB supergravity. The two and three-point functions of chiral primary operators are protected against radiative corrections [3], hence the scaling dimensions are given by $\Delta = k$ exactly. And indeed these free field theory correlation functions can be matched to the dual supergravity result [4]. Since 2002 the interest has advanced to the true string theoretical domain of the correspondence in which string excitations of the $AdS_5 \times S^5$-theory are matched to generic local gauge theory operators with nontrivial two-point functions, i.e. anomalous scaling dimensions. Here one is close to a complete solution of the problem of determining the spectrum of scaling dimensions in the theory, determined by a set of Bethe equations [5] (see [6] for reviews). In parallel and connected to this, our understanding of scattering amplitudes in the theory has increased considerably. Here a novel dual superconfromal symmetry of the theory in momentum space enabled one to find the all-loop form of maximally helicity violating gluon four and five-point amplitudes [7]. Moreover, these amplitudes are dual to light-like Wilson loops (see [8] for a reviews).

Despite the protectedness of two and three-point functions of $\mathcal{N} = 4$ super Yang-Mills chiral primaries, the $n \geq 4$ point functions are in general highly non-trivial functions of the 't Hooft coupling constant λ and $1/N$. A lot of work has been devoted to the study of four-point functions, both from the gauge theory side up to two-loop order and in the supergravity approximation [9–20].[1] An important structural insight has been the universal factorization of the quantum corrections to the correlation function of four chiral primaries of weight k into a universal prefactor $\mathcal{R}_{\mathcal{N}=4}$ and a non-universal remainder [16]

$$\langle \mathcal{O}_k(x_1)\, \mathcal{O}_k(x_3)\, \mathcal{O}_k(x_3)\, \mathcal{O}_k(x_4)\, \rangle_{\text{quant}} = \mathcal{R}_{\mathcal{N}=4}(s,t) \cdot \mathcal{F}^{(k)}(s,t,\lambda)\,, \tag{1.1}$$

with conformal cross-ratios s and t. This factorization was shown to arise both at weak and strong coupling, as well as non-perturbatively in instanton computations [22]. The prefactor $\mathcal{R}_{\mathcal{N}=4}$ was analyzed also in [19, 20], where it was related to superconformal Ward-Takahashi identities satisfied by the 4-point correlation functions.

In this note we wish to modestly extend these structural insights to $n \geq 4$ point functions of chiral primaries by performing a diagramatical computation at one-loop order in the planar limit. The result of this paper is a universal form for the n-point function at one-loop detailed in equation (3.25): The one-loop contribution may be rewritten as a sum over certain tree-level amplitudes with the topology of the disc where 4 insertions are at the boundary and $(n-4)$ insertions are situated in the bulk, multiplied by the one-loop scalar box integral. The factorization with $\mathcal{R}_{\mathcal{N}=4}$ for $n = 4$ of equation (1.1) at one-loop order is an immediate corrollary of our result. In essence our result is a consequence of the fact that the one-loop interactions involve at most four points and that the conformal symmetry does not allow for the appearance of non-trivial functions depending on only three space-time points.

Furthermore, we apply this result to compute a selection of five and six-point functions. The structures we find suggest that the one-loop corrections to a general five-point function

[1] It should be noted that there exists a very specific class of so called extremal and next-to-extremal n-point functions, which are characterized by the property of having tree-level diagrams factorized into two and three-point functions. These $(n \geq 4)$-point functions do not receive one-loop corrections [21] as well.

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may be decomposed into the two minimal four-point and five-point functions of weight two operators multiplied by tree-level contractions.

The motivation for these considerations comes from at least two viewpoints: In a companion paper [23], we study examples of n-point functions of chiral primary operators who have common supersymmetries. We apply the results of this paper to evaluate the one-loop corrections to these n-point functions and find that the radiative corrections vanish. Beyond this specific application, we feel that following the tremendous advances on our understanding of $n = 2$ point functions in the $\mathcal{N} = 4$ gauge theory upon exploiting integrability, the time is ripe to turn one's attention to the cases with $n \geq 3$. We hope that our result will prove useful for such an endeavour in the future.

2 Notation

A chiral primary operator is a composite scalar field of dimension k being the lowest component of a 1/2 BPS multiplet. In terms of the elementary fields of $\mathcal{N} = 4$ super Yang-Mills (ϕ^I, A_μ, ψ^a) it is given by $\mathrm{Tr}[\,\phi^{\{I_1} \dots \phi^{I_k\}}\,]$ with $I = 1, \dots, 6$ and $\{\dots\}$ denoting traceless symmetrization. A very convenient way to handle the SO(6) indices is to represent the chiral primary with the help of a complex null vector u^I (with $u^I\, u^I = 0$ and $u^I\, \bar{u}^I = 1$) (see e.g. [16])

$$\mathcal{O}^u_k(x) := u^{I_1} \dots u^{I_k}\ \mathrm{Tr}\left[\phi^{I_1}(x) \dots \phi^{I_k}(x)\right] = \mathrm{Tr}\left[(u \cdot \phi(x))^k\right]. \tag{2.1}$$

We shall be interested in computing n-point correlation functions of these operators

$$\left\langle \mathcal{O}^{u_1}_{k_1}(x_1)\, \mathcal{O}^{u_2}_{k_2}(x_2)\, \mathcal{O}^{u_3}_{k_3}(x_3) \dots \mathcal{O}^{u_n}_{k_n}(x_n) \right\rangle \tag{2.2}$$

at one-loop. We furthermore define the tree level contraction of two scalars[2]

$$[ij] := \langle (u_i \cdot \phi(x_i))\, (u_j \cdot \phi(x_j)) \rangle_{\mathrm{tree}} = \frac{u_i \cdot u_j}{(2\pi)^2\, x_{ij}^2}\,, \qquad\qquad x_{ij} := x_i - x_j\,. \tag{2.3}$$

Moreover, for any choice of four operators there are two conformally invariant cross-ratios, which will show up in the calculation. Focusing on the four-point function, we define s and t as the cross-ratios of (x_1, x_2, x_3, x_4). They may also be expressed via one complex number μ

$$s = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = \mu\, \bar{\mu} \qquad\qquad t = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = (1-\mu)\,(1-\bar{\mu})\,. \tag{2.4}$$

With the definitons

$$\mathcal{X} = [12][34]\,, \qquad \mathcal{Y} = [13][24]\,, \qquad \mathcal{Z} = [14][23]\,. \tag{2.5}$$

the universal polynomial prefactor of equation (1.1) may be written in two ways as

$$\begin{aligned} \mathcal{R}_{\mathcal{N}=4} &= s\,(\mathcal{Y}-\mathcal{X})\,(\mathcal{Z}-\mathcal{X}) + t\,(\mathcal{Z}-\mathcal{X})\,(\mathcal{Z}-\mathcal{Y}) + (\mathcal{Y}-\mathcal{X})\,(\mathcal{Y}-\mathcal{Z}) \\ &= \big(\mu\,(\mathcal{X}-\mathcal{Z}) + \mathcal{Z} - \mathcal{Y}\big)\,\big(\bar{\mu}\,(\mathcal{X}-\mathcal{Z}) + \mathcal{Z} - \mathcal{Y}\big)\,. \end{aligned} \tag{2.6}$$

Such factorization was also observed in [19, 20].

[2] Since we consider only planar diagrams, we suppress throughout the gauge group indices. Some of our considerations can be applied also to non-planar graphs.

3 The perturbative computation

We present here some basic formulas that are helpful in order to address the one-loop radiative corrections to correlation functions of local operators built from scalars.

Following [24] we introduce the scalar propagator and some fundamental tree functions in configuration space

$$I_{12} = \frac{1}{(2\pi)^2(x_1 - x_2)^2}, \tag{3.1}$$

$$Y_{123} = \int d^4w\, I_{1w} I_{2w} I_{3w}, \tag{3.2}$$

$$X_{1234} = \int d^4w\, I_{1w} I_{2w} I_{3w} I_{4w}, \tag{3.3}$$

$$H_{12,34} = \int d^4u\, d^4v\, I_{1u} I_{2u} I_{uv} I_{3v} I_{4v}. \tag{3.4}$$

We have put the space-time points as indices to the function to make the expressions more compact. These functions are all finite except in certain limits. For example Y, X and H diverge logarithmically when $x_1 \to x_2$. The functions X and Y can be evaluated explicitly [25]

$$X_{1234} = \frac{\pi^2 \Phi(s,t)}{(2\pi)^8 (x_1 - x_3)^2 (x_2 - x_4)^2}, \tag{3.5}$$

$$Y_{123} = \lim_{x_4 \to \infty} (2\pi)^2\, x_4^2\, X_{1234}. \tag{3.6}$$

In the euclidean region ($\sqrt{s} + \sqrt{t} \geq 1$, $\left|\sqrt{s} - \sqrt{t}\right| \leq 1$) the function $\Phi(s,t)$ can be written in a manifestly real fashion as

$$\begin{aligned}
\Phi(s,t) =& \frac{1}{A}\,\mathrm{Im}\left(\mathrm{Li}_2 \frac{e^{i\varphi}\sqrt{s}}{\sqrt{t}} + \ln\frac{\sqrt{s}}{\sqrt{t}}\,\ln\frac{\sqrt{t} - e^{i\varphi}\sqrt{s}}{\sqrt{t}}\right) \\
e^{i\varphi} =& i\sqrt{-\frac{1 - s - t - 4iA}{1 - s - t + 4iA}}, \qquad\qquad A = \frac{1}{4}\sqrt{4st - (1 - s - t)^2}.
\end{aligned} \tag{3.7}$$

It is positive everywhere, vanishes only in the limit $s, t \to \infty$ and has the hidden symmetry $\Phi(s,t) = \Phi(1/s, t/s)/s$.

There seems to be no analytic expression for the function H. However, the Feynman rules lead to its appearance in the combination which can be expressed in terms of the functions X and Y [12, 24]

$$F_{12,34} = \frac{(\partial_1 - \partial_2)\cdot(\partial_3 - \partial_4) H_{12,34}}{I_{12} I_{34}} = \frac{X_{1234}}{I_{13} I_{24}} - \frac{X_{1234}}{I_{14} I_{23}} + G_{1,34} - G_{2,34} + G_{3,12} - G_{4,12}\,, \tag{3.8}$$

$$G_{1,34} = \frac{Y_{134}}{I_{14}} - \frac{Y_{134}}{I_{13}}. \tag{3.9}$$

3.1 Combining the basic interactions

In order to simplify the one-loop perturbative computation of n-point functions of chiral primaries of $\mathcal{N} = 4$ super Yang-Mills we shall develop a number of insertion formulas, which allow one to easily construct the one-loop corrections to a given tree level graph.

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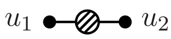
We work with scalar propagators normalized to

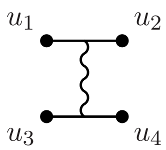
$$[12] := u_1^{I_1}\, u_2^{I_2}\, \big\langle \phi^{I_1}(x_1)\, \phi^{I_2}(x_2) \big\rangle_{\text{tree level}} =$$  $$= (u_1 \cdot u_2)\, I_{12}\,. \tag{3.10}$$

As mentioned, we do not record SU(N) factors.

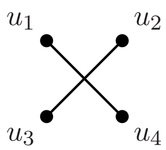
Using this language we then note the following one-loop planar insertion formulas



$$= -\lambda\, (u_1 \cdot u_2)\, I_{12}\, \frac{Y_{112} + Y_{122}}{I_{12}} \tag{3.11}$$



$$= \frac{\lambda}{2}\, (u_1 \cdot u_2)\, (u_3 \cdot u_4)\, I_{12}\, I_{34}\, F_{12,34} \tag{3.12}$$



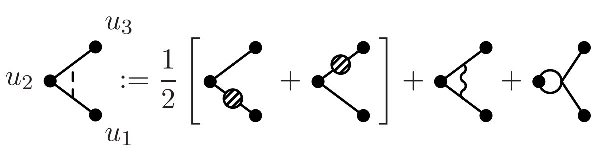
$$= \frac{\lambda}{2} \Big[2\, (u_2 \cdot u_3)\, (u_1 \cdot u_4) - (u_2 \cdot u_4)\, (u_1 \cdot u_3) - (u_1 \cdot u_2)\, (u_3 \cdot u_4) \Big]\, X_{1234} \tag{3.13}$$

where the grey blob stands for the one-loop self-energy and curly lines denote gluon propagators. We furthermore note the relevant pinching limits of the functions defined in equations (3.1)-(3.4) in point-splitting regularization

$$\begin{aligned}
Y_{112} &= Y_{122} = -\frac{1}{16\pi^2} \left(\ln \frac{\epsilon^2}{x_{12}^2} - 2 \right) I_{12} \\
X_{1123} &= -\frac{1}{16\pi^2}\, I_{12}\, I_{13} \left(\ln \frac{\epsilon^2\, x_{23}^2}{x_{12}^2\, x_{31}^2} - 2 \right) \\
F_{12,13} &= -\frac{1}{16\pi^2} \left(\ln \frac{\epsilon^2}{x_{23}^2} - 2 \right) + Y_{123} \left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}} \right) \\
X_{1122} &= -\frac{1}{8\pi^2}\, I_{12}^2 \left(\ln \frac{\epsilon^2}{x_{12}^2} - 1 \right) \\
F_{12,12} &= -\frac{1}{8\pi^2}\, I_{12}^2 \left(\ln \frac{\epsilon^2}{x_{12}^2} - 3 \right) .
\end{aligned} \tag{3.14}$$

The strategy for the computation of one-loop corrections to higher point functions of chiral primary operators $\mathcal{O}_k^{u_i}(x) = \text{Tr}[(u_i^I\, \phi^I(x))^k]$ will be to consider the "dressing" of the tree-level graphs. We shall distinguish the following interaction insertions types:

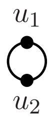
(i) Corner interactions. Combining one-half of the self-energy corrections on every leg with the gluon exchange and four-point interaction one has for every corner

 $$= C_{123} [12][23]\,. \tag{3.15}$$

where we have defined the corner insertion

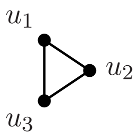
$$C_{123} := \frac{\lambda}{2}\, Y_{123} \left(\frac{1}{I_{12}} + \frac{1}{I_{23}} - \frac{2}{I_{31}} \right) . \tag{3.16}$$

Using this result and the pinching identities (3.14) it is easy to check the vanishing of the one-loop correction to the two-point functions $\langle\, \mathcal{O}_2(x_1)\, \mathcal{O}_2(x_2)\, \rangle$



$$\Big|_{\substack{\text{1-loop}\\ \text{dressed}}} = 2^2\, (C_{212} + C_{121})\, [12]^2 = 0\, . \tag{3.17}$$

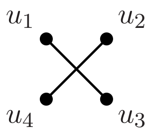
Similarly one shows the vanishing of the one-loop correction to the three-point functions $\langle\, \mathcal{O}_2(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_2(x_3)\, \rangle$



$$\Big|_{\substack{\text{1-loop}\\ \text{dressed}}} = 2^3\, (C_{123} + C_{231} + C_{312})\, [12][23][31] = 0\, . \tag{3.18}$$

(ii) Non-corner interactions. Here we have to consider the gluon exchange and scalar four-point graphs of (3.12) and (3.13). These graphs involve the invariant cross-ratios and the scalar graph also includes a contraction which at tree-level would not be planar: The crossed contraction [14][23] in the numbering convention of (3.13). From now on, however, we shall switch to a cyclic ordering of the points $\{1, 2, 3, 4\}$ and define

$$S_{1234} :=$$

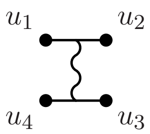


$$= \frac{\lambda}{2}\, \frac{X_{1234}}{I_{13} I_{24}} \big(2\, [13][24] - t\, [23][14] - s\, [12][34] \big)\, , \tag{3.19}$$

where we have used the cross-ratios $s = \frac{I_{13} I_{24}}{I_{12} I_{34}}$ and $t = \frac{I_{13} I_{24}}{I_{14} I_{23}}$ of (2.4).

For the gluon exchange graph (3.12) we define

$$G_{1234} :=$$



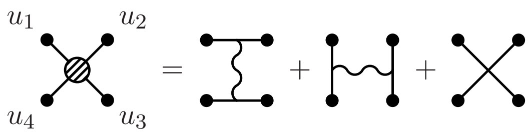
$$= \frac{\lambda}{2}\, F_{12,43}\, [12][34] = \left[\frac{\lambda}{2}\, \frac{X_{1234}}{I_{13} I_{24}} \big(t - 1 \big) + C_{12,43} \right] [12][34] \tag{3.20}$$

$$C_{12,43} := \frac{1}{3} \big(C_{123} + C_{412} + C_{341} + C_{234} - C_{124} - C_{243} - C_{431} - C_{312} \big)\, . \tag{3.21}$$

Note that this expression does preserve the structure of the tree-level contractions. In the final expression in (3.20) we wrote $F_{12,43}$ as a sum of two terms, the first involving the box integral $\Phi(s, t)$ (3.7) and the second which is a sum of terms identical to the corner graphs (3.15) and which does not involve the cross-ratios.

Lastly we define a full four-point planar interaction insertion by combining both orientations and removing the corner interactions

$$D_{1234} :=$$



$$- C_{12,43}[12][34] - C_{14,23}[14][23]$$

$$\begin{aligned} &= \frac{\lambda}{2}\frac{X_{1234}}{I_{13}I_{24}}\big(2\,[13][24] + (s-1-t)[14][23] + (t-1-s)\,[12][34]\big) \\ &= \frac{\lambda}{32\pi^2}\,\Phi(s,t)\,\big(2\,[13][24] + (s-1-t)[14][23] + (t-1-s)\,[12][34]\big) \end{aligned} \tag{3.22}$$

Note that D_{1234} has manifest cyclic symmetry as well as reflection symmetry $2 \leftrightarrow 4$ (which is accompanied, of course, by $s \leftrightarrow t$).

3.2 Cancelation of corner graphs

We would like to calculate the full connected one-loop planar interacting diagrams. In order to enumerate these we can relate them to tree-level diagrams by cutting the interacting line. Cutting the dashed line in the combined corner interaction (3.15) gives a single tree-level graph. The same is true for the four-point gluon exchange (3.20), but not for the four-scalar vertex (3.19) as it can be cut in two different ways. One point to note, though, is that the underlying tree-level graph of a non-corner interaction may be disconnected.

This allows us therefore to go in the opposite direction, starting with all tree-level planar graphs (including disconnected ones) and dress them up with one-loop interactions. This dressing involves the exchange of gluons (and also the scalar interaction) between pairs of lines which form the border of a face on the graph.

Consider such a face with n vertices at positions $x_1, \dots x_n$ cyclically ordered and with the local operators at x_i formulated by contracting the scalar fields with the six-vectors u_i^I. At one-loop there will be the following corner interactions within this face

$$\sum_{i=1}^{n} C_{(i-1)\,i\,(i+1)}\,, \qquad i_0 \equiv i_n\,. \tag{3.23}$$

In addition there will be four-point interactions from all pairs of non-adjacent links, where we strip away the tree-level contractions. Let us now consider the sum of the corner interactions and the corner-like contributions $C_{ij,lm}$ to the four-point interactions

$$\begin{aligned} \sum_{i=1}^{n}\Big(C_{i-1\,i\,i+1} + \frac{1}{2}\sum_{j\neq i-1,i,i+1} C_{i\,i+1,j\,j+1}\Big) = \sum_{i=1}^{n}\Bigg(C_{i-1\,i\,i+1} - \frac{1}{6}\sum_{j=i+2}^{i-2}\big(C_{i\,i+1\,j} + C_{i+1\,j\,j+1} \\ + C_{j\,j+1\,i} + C_{j+1\,i\,i+1} - C_{i\,i+1\,j+1} - C_{i+1\,i\,j} - C_{j\,j+1\,i+1} - C_{j+1\,j\,i}\big)\Bigg) = 0\,. \end{aligned} \tag{3.24}$$

To see this cancelation it is convenient to rearrange the sum by the differences of indices, noting that $C_{ijl} = C_{lji}$. Lastly one needs the relation $C_{ijl} + C_{jli} + C_{lij} = 0$ (3.18).

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$$\left\langle \mathcal{O}^{u_i}_{k_i-1}\mathcal{O}^{u_j}_{k_j-1}\mathcal{O}^{u_l}_{k_l-1}\mathcal{O}^{u_m}_{k_m-1}\middle| \prod_{p\neq i,j,l,m} \mathcal{O}^{u_p}_{k_p} \right\rangle_{\text{tree, disc}} =$$

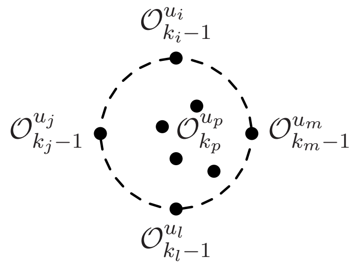

Figure 1. Graphical representation of the disc correlation function of equation (3.25). The first four operators are placed on the boundary of the disc and all the others are in the interior. In (3.25) one is instructed to sum over all planar tree-level contractions of these operators.

3.3 General one-loop insertion formula

Having proved that all corner interactions cancel we can write down the general one-loop four-point insertion formula. Starting with any planar tree graph we dress up every pair of links that are on one face but not adjacent with the basic four-point insertion $\frac{1}{2}D_{ijlm}$ (3.22). The factor of 1/2 is there because the same insertion will come from another tree graph with the contraction $[ij][lm]$ replaced by $[jl][mi]$.

We get the general one-loop result

$$\left\langle \mathcal{O}^{u_1}_{k_1}\cdots\mathcal{O}^{u_n}_{k_n}\right\rangle_{\text{1-loop}} = \sum_{i,j,l,m} k_ik_jk_lk_m\, D_{ijlm}\left\langle \mathcal{O}^{u_i}_{k_i-1}\mathcal{O}^{u_j}_{k_j-1}\mathcal{O}^{u_l}_{k_l-1}\mathcal{O}^{u_m}_{k_m-1}\middle| \prod_{p\neq i,j,l,m} \mathcal{O}^{u_p}_{k_p} \right\rangle_{\text{tree, disc}} \tag{3.25}$$

The expression on the right-hand side is the tree level n-point function of the operators where one scalar is removed from each of the four operators and this should be a planar diagram with the topology of a disk, with the four operators with labels $ijlm$ inserted (in this order) at the boundary of the disc and the rest in the bulk. The contractions represented by D_{ijlm} are then outside of the disc, but still have a planar spherical topology. This amplitude is represented pictorially in figure 1.

D_{ijlm} (3.22) involves a transcendental function $\Phi(s,t)$ (3.7) of the cross-ratios of the four points x_i, x_j, x_l and x_m. Since the tree-level contraction among these four operators and all the others involves only rational functions, it is natural to separate the graphs in this way. When looking for special cancelations in the one-loop amplitudes at generic positions (as done in [23]) there are no algebraic relations among the D-functions. The exception are terms with the same boundary vertices i, j, k and l, but in a different order, since the functions D_{ijlm} are then related to each-other. Due to the cyclic and reflection symmetry of D_{ijlm}, there are only three ineqivalent orderings $ijlm$, $iljm$ and $ijml$.

The crucial relation we use extensively is

$$D_{ijlm} + D_{iljm} + D_{ijml} = 0\,. \tag{3.26}$$

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4 Four-point functions

We would like here to rederive the factorization formula of the four-point function (1.1) of [16] at one-loop using the general formula (3.25). It is instructive to start with the simplest four-point function, of four operators of dimension two.

4.1 Four-point functions of $\mathcal{O}_2^{u_i}$

Applying our general formula (3.25) in this case we have

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\rangle_{\text{1-loop}} = 16\big(&D_{1234}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}\rangle_{\text{tree, disc}} \\ &+D_{1324}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\rangle_{\text{tree, disc}} + D_{1243}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\mathcal{O}_1^{u_3}\rangle_{\text{tree, disc}}\big).\end{aligned} \tag{4.1}$$

For a given ordering there are two planar tree diagrams on the disc, with a pair of contractions labeled $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ of (2.5). We get

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\rangle_{\text{1-loop}} &= 16\big(D_{1234}(\mathcal{X}+\mathcal{Z}) + D_{1243}(\mathcal{Y}+\mathcal{X}) + D_{1324}(\mathcal{Z}+\mathcal{Y})\big) \\ &= -16\big(D_{1234}\mathcal{Y} + D_{1243}\mathcal{Z} + D_{1324}\mathcal{X}\big).\end{aligned} \tag{4.2}$$

To get the last line we subtracted from all the terms the sum of all three pair-wise contractions $\mathcal{X}+\mathcal{Y}+\mathcal{Z}$ and used the fact that $(D_{1234}+D_{1243}+D_{1324}) = 0$ (3.26). So the result is written as minus the sum of all *non*-planar contractions.

Now we note that we can also express D_{1234} of (3.22) in terms of the pair-wise contractions and the universal polynomial prefactor $\mathcal{R}_{\mathcal{N}=4}$ of (2.6) as

$$D_{1234} = \frac{\lambda}{2}\,\frac{X_{1234}}{I_{13}I_{24}}\big(2\mathcal{Y} - \mathcal{X} - \mathcal{Z} + (s-t)(\mathcal{Z}-\mathcal{X})\big) = \frac{\lambda}{2}\,\frac{\Phi(s,t)}{16\pi^2}\,\frac{\partial \mathcal{R}_{\mathcal{N}=4}}{\partial \mathcal{Y}}\,, \tag{4.3}$$

and likewise for the other ones. This gives

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\rangle_{\text{1-loop}} &= -\frac{\lambda}{2\pi^2}\,\Phi(s,t)\left(\mathcal{Y}\frac{\partial \mathcal{R}_{\mathcal{N}=4}}{\partial \mathcal{Y}} + \mathcal{Z}\frac{\partial \mathcal{R}_{\mathcal{N}=4}}{\partial \mathcal{Z}} + \mathcal{X}\frac{\partial \mathcal{R}_{\mathcal{N}=4}}{\partial \mathcal{X}}\right) \\ &= -\frac{\lambda}{\pi^2}\,\Phi(s,t)\,\mathcal{R}_{\mathcal{N}=4}\end{aligned} \tag{4.4}$$

4.2 General four-point function

Now consider a general four-point function

$$\left\langle \mathcal{O}_{k_1}^{u_1}\mathcal{O}_{k_2}^{u_2}\mathcal{O}_{k_3}^{u_3}\mathcal{O}_{k_4}^{u_4}\right\rangle. \tag{4.5}$$

Applying our general formula (3.25) we have

$$\begin{aligned}\left\langle \mathcal{O}_{k_1}^{u_1}\mathcal{O}_{k_2}^{u_2}\mathcal{O}_{k_3}^{u_3}\mathcal{O}_{k_4}^{u_4}\right\rangle_{\text{1-loop}} = k_1k_2k_3k_4\Big(&D_{1234}\left\langle \mathcal{O}_{k_1-1}^{u_1}\mathcal{O}_{k_2-1}^{u_2}\mathcal{O}_{k_3-1}^{u_3}\mathcal{O}_{k_4-1}^{u_4}\right\rangle_{\text{tree, disc}} \\ &+D_{1324}\left\langle \mathcal{O}_{k_1-1}^{u_1}\mathcal{O}_{k_3-1}^{u_3}\mathcal{O}_{k_2-1}^{u_2}\mathcal{O}_{k_4-1}^{u_4}\right\rangle_{\text{tree, disc}} \\ &+D_{1243}\left\langle \mathcal{O}_{k_1-1}^{u_1}\mathcal{O}_{k_2-1}^{u_2}\mathcal{O}_{k_4-1}^{u_4}\mathcal{O}_{k_3-1}^{u_3}\right\rangle_{\text{tree, disc}}\Big).\end{aligned} \tag{4.6}$$

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Let us examine the term multiplying D_{1234}. It involves all possible tree-level planar contractions on the disc. It is clear that it can be factorized as

$$\left\langle \mathcal{O}^{u_1}_{k_1-1}\mathcal{O}^{u_2}_{k_2-1}\mathcal{O}^{u_3}_{k_3-1}\mathcal{O}^{u_4}_{k_4-1}\right\rangle_{\text{tree, disc}} = \big([12][34]+[14][23]\big)\times\{\cdots\} \tag{4.7}$$

where $\{\cdots\}$ stands for some planar tree-level contractions of $\mathcal{O}^{u_1}_{k_1-2}\mathcal{O}^{u_2}_{k_2-2}\mathcal{O}^{u_3}_{k_3-2}\mathcal{O}^{u_4}_{k_4-2}$.

The non-trivial fact about the factorization formula of Arutyunov et al. [16, 17], is that exactly the same combinatorics of tree-contractions appear in all three permutations of the order of insertions in (4.6). Writing the pairwise contractions as $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ (2.5) the four point function becomes

$$\left\langle \mathcal{O}^{u_1}_{k_1}\mathcal{O}^{u_2}_{k_2}\mathcal{O}^{u_3}_{k_3}\mathcal{O}^{u_4}_{k_4}\right\rangle_{\text{1-loop}} = \tag{4.8}$$

$$= k_1k_2k_3k_4\big(D_{1234}(\mathcal{X}+\mathcal{Z})+D_{1243}(\mathcal{Y}+\mathcal{X})+D_{1324}(\mathcal{Z}+\mathcal{Y})\big)\{\cdots\}$$

$$= -\frac{k_1k_2k_3k_4}{16}\frac{\lambda}{\pi^2}\,\Phi(s,t)\,\mathcal{R}_{\mathcal{N}=4}\,\{\cdots\}\,. \tag{4.9}$$

Indeed when $k_1=k_2=k_3=k_4$, a simple inspection reveals that the same combinatorics appear in all the terms in (4.6).

5 Five-point functions

5.1 Five-point functions of $\mathcal{O}^{u_i}_2$

We turn now to the simplest five-point function, that of five operators of dimension two. Applying our general formula (3.25) in this case we have

$$\begin{aligned}
\langle \mathcal{O}^{u_1}_2\mathcal{O}^{u_2}_2\mathcal{O}^{u_3}_2\mathcal{O}^{u_4}_2\mathcal{O}^{u_5}_2\rangle_{\text{1-loop}} = 16\,\big(&D_{1234}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_4}_1|\mathcal{O}^{u_5}_2\rangle_{\text{tree, disc}}\\
&+D_{1324}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_4}_1|\mathcal{O}^{u_5}_2\rangle_{\text{tree, disc}}+D_{1243}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_4}_1\mathcal{O}^{u_3}_1|\mathcal{O}^{u_5}_2\rangle_{\text{tree, disc}}\big)\\
&+16\,\big(D_{1235}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_5}_1|\mathcal{O}^{u_4}_2\rangle_{\text{tree, disc}}+\cdots\big)\\
&+16\,\big(D_{1245}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_4}_1\mathcal{O}^{u_5}_1|\mathcal{O}^{u_3}_2\rangle_{\text{tree, disc}}+\cdots\big)\\
&+16\,\big(D_{1345}\,\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_4}_1\mathcal{O}^{u_5}_1|\mathcal{O}^{u_2}_2\rangle_{\text{tree, disc}}+\cdots\big)\\
&+16\,\big(D_{2345}\,\langle \mathcal{O}^{u_2}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_4}_1\mathcal{O}^{u_5}_1|\mathcal{O}^{u_1}_2\rangle_{\text{tree, disc}}+\cdots\big)\,.
\end{aligned} \tag{5.1}$$

The function D_{ijlm} is given in (3.22). To evaluate (5.1) we need to find the tree level disc amplitudes, which are all the same, up to permutations of indices. A simple application of Wick's theorem gives

$$\langle \mathcal{O}^{u_1}_1\mathcal{O}^{u_2}_1\mathcal{O}^{u_3}_1\mathcal{O}^{u_4}_1|\mathcal{O}^{u_5}_2\rangle_{\text{tree, disc}} = 2\,\big([12,34|5]+[14,23|5]\big) \tag{5.2}$$

where we define the pair-wise contraction through a fifth point as a generalization of $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ (2.5)

$$[12,34|5] = [15][25][34]+[12][35][45]\,. \tag{5.3}$$

As in the case of the four-point function in section 4, we subtract the sum of all three possible contractions $([12,34|5]+[13,24|5]+[14,23|5])$ from the first three terms of (5.1) using the fact that $(D_{1234} + D_{1243} + D_{1324}) = 0$. After similar manipulations of all the other terms, this allows us to express (5.1) as minus the sum over all non-planar graphs

$$\begin{aligned}\langle \mathcal{O}_2^{u_1} \cdots \mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} = & -32\left(D_{1234}[13,24|5] + D_{1324}[12,34|5] + D_{1243}[14,23|5]\right) \\ & -32\left(D_{1235}[13,25|4] + D_{1325}[12,53|4] + D_{1253}[15,23|4]\right) \\ & -32\left(D_{1254}[15,24|3] + D_{1524}[12,45|3] + D_{1245}[14,25|3]\right) \\ & -32\left(D_{1534}[13,54|2] + D_{1354}[15,34|2] + D_{1543}[14,53|2]\right) \\ & -32\left(D_{5234}[53,24|1] + D_{5324}[52,34|1] + D_{5243}[54,23|1]\right) \end{aligned} \tag{5.4}$$

5.2 Five-point functions of four $\mathcal{O}_2^{u_i}$ and one $\mathcal{O}_4^{u_5}$

Next we look at the five-point function with four operators of dimension two and one of dimension four. Applying our general formula (3.25) in this case we have

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_4^{u_5}\rangle_{\text{1-loop}} = & \; 16\big(D_{1234}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}|\mathcal{O}_4^{u_5}\rangle_{\text{tree, disc}} \\ & +D_{1324}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}|\mathcal{O}_4^{u_5}\rangle_{\text{tree, disc}} + D_{1243}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\mathcal{O}_1^{u_3}|\mathcal{O}_4^{u_5}\rangle_{\text{tree, disc}}\big) \\ & +32\left(D_{1235}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_3^{u_5}|\mathcal{O}_2^{u_4}\rangle_{\text{tree, disc}} + \cdots\right) \\ & +32\left(D_{1245}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_4}\mathcal{O}_3^{u_5}|\mathcal{O}_2^{u_3}\rangle_{\text{tree, disc}} + \cdots\right) \\ & +32\left(D_{1345}\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}\mathcal{O}_3^{u_5}|\mathcal{O}_2^{u_2}\rangle_{\text{tree, disc}} + \cdots\right) \\ & +32\left(D_{2345}\,\langle \mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}\mathcal{O}_3^{u_5}|\mathcal{O}_2^{u_1}\rangle_{\text{tree, disc}} + \cdots\right). \end{aligned} \tag{5.5}$$

The first tree-level disc amplitude in this expression is

$$\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}|\mathcal{O}_4^{u_5}\rangle_{\text{tree, disc}} = 4\,[15][25][35][45]\,. \tag{5.6}$$

this is clearly independent of the order of the labels 1234, so the sum of the first three terms in (5.5) is proportional to $(D_{1234} + D_{1324} + D_{1243})$, which vanishes by (3.26).

The other D_{ijlm} multiply other tree-level amplitudes. For example D_{1235} multiplies the term

$$\begin{aligned}\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_3^{u_5}|\mathcal{O}_2^{u_4}\rangle_{\text{tree, disc}} = 2\Big(& [35][15][45][24] + [25][35][45][14] + [25][15][45][34] \\ & +2[45]^2\left([35][12] + [15][23]\right)\Big). \end{aligned} \tag{5.7}$$

Now we note that under permutations of 1, 2 and 3 the terms in the first line get interchanged. Therefore these will multiply the sum $(D_{1235} + D_{1325} + D_{1253})$ and hence vanish (recall that $D_{1253} = D_{2135}$). We are left with the terms on the second line, which may be written as $2[45]^2\,\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_5}\rangle_{\text{tree, disc}}$, and the sum over the three permutations is proportional to the one-loop four-point function

$$[45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} \tag{5.8}$$

The five-point function (5.5) is therefore a sum over four terms, each proportional to a one-loop four-point function

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_4^{u_5}\rangle_{\text{1-loop}} = & \; 8[15]^2\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} + 8[25]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} \\ & +8[35]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} + 8[45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}}\,. \end{aligned} \tag{5.9}$$

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The reduction of this near-extremal correlator to weight two four-point function was also observed in [26].

5.3 Five-point functions of three $\mathcal{O}_2^{u_i}$ and two $\mathcal{O}_3^{u_i}$

The next five-point function we evaluate is

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_3^{u_5}\rangle_{\text{1-loop}} \tag{5.10}$$

Like in the last example we have to consider a few inequivalent disc amplitudes to calculate (3.25). First there are the amplitudes multiplying D_{1234} and D_{1235} with different orderings which are of the form

$$\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_2^{u_4}|\mathcal{O}_3^{u_5}\rangle_{\text{tree, disc}} = 3\,[45]\big([12,34|5]+[14,23|5]\big)\,. \tag{5.11}$$

These are proportional to the terms appearing in the one-loop correction to the simplest five-point function (5.2).

Then, when the interaction is among the points 1245, there are two different inequivalent orderings of the vertices, when 4 and 5 are adjacent or not. In the first case we have

$$\begin{aligned}\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}|\mathcal{O}_2^{u_3}\rangle_{\text{tree, disc}} &= 2\Big([14][24][35]^2+[15][25][34]^2\\ &\quad+[34][35]\big([12][45]+[15][24]\big)+[45]\big([12,45|3]+[15,24|3]\big)\Big)\end{aligned} \tag{5.12}$$

and likewise permuting $1\leftrightarrow 2$

$$\begin{aligned}\langle \mathcal{O}_1^{u_2}\mathcal{O}_1^{u_1}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}|\mathcal{O}_2^{u_3}\rangle_{\text{tree, disc}} &= 2\Big([14][24][35]^2+[15][25][34]^2\\ &\quad+[34][35]\big([12][45]+[25][14]\big)+[45]\big([12,45|3]+[14,25|3]\big)\Big)\end{aligned} \tag{5.13}$$

For the third, the inequivalent, ordering we have

$$\begin{aligned}\langle \mathcal{O}_1^{u_1}\mathcal{O}_2^{u_4}\mathcal{O}_1^{u_2}\mathcal{O}_2^{u_5}|\mathcal{O}_2^{u_3}\rangle_{\text{tree, disc}} &= 2\Big([14][24][35]^2+[15][25][34]^2\\ &\quad+[34][35]\big([14][25]+[15][24]\big)+[45]\big[14,25|3]+[15,24|3]\big)\Big).\end{aligned} \tag{5.14}$$

Now we note that $[14][24][35]^2$ and $[15][25][34]^2$ appear in each of the three permutations, and therefore vanish.

As before, by subtracting the sum of the contractions appearing above we get the simple expression for the terms multiplying D_{1245} and its permutations

$$\begin{aligned}&-[34][35]\big(D_{1245}[14][25]+D_{2145}[15][24]+D_{1425}[12][45]\big)\\ &-[45]\big(D_{1245}[14,25|3]+D_{2145}[15,24|3]+D_{1425}[12,45|3]\big)\end{aligned} \tag{5.15}$$

Note that the three terms on the first line are proportional to the one-loop four-point function of operators of dimension two (4.2). The terms on the second line are like those appearing in the expression for the simplest five-point function (5.4).

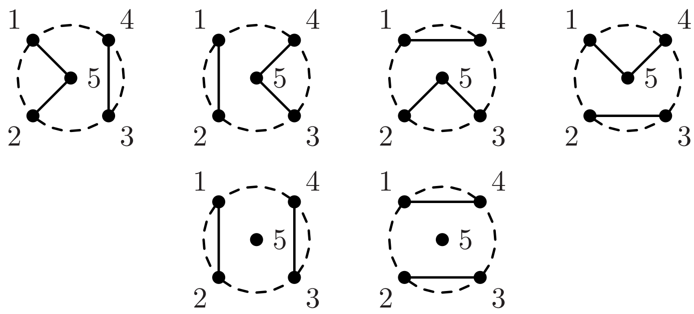

Figure 2. The four different planar contractions on the disc passing once through the fifth point (top line), and the two not passing through it (bottom line).

Summing all the different terms gives the one-loop five-point amplitude as a sum over the simplest four-point function and the simplest five-point function

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_3^{u_5}\rangle_{\text{1-loop}} = \frac{9}{4}\Big([45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} \\ +2\,[41][15]\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} +2\,[42][25]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}} \\ +2\,[43][35]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle_{\text{1-loop}}\Big). \qquad (5.16)\end{aligned}$$

5.4 General five-point function

We do not have a general explicit formula for the one-loop five-point function, but beyond the three examples above we calculated eleven more examples of five-point functions with operators of total dimension up to sixteen. They are listed in appendix A.

Starting with our general insertion formula (3.25), and focusing on the terms where the interaction is among the points x_1, x_2, x_3 and x_4, one can examine all the remaining tree-level planar contractions on the disc. As in the case of the general four-point function it is clear that D_{1234} will always show up in combinations of the form (see figure 2)

$$D_{1234}\big([12][34]+[14][23]\big)\,,\quad D_{1234}\big([12,34|5]+[14,23|5]\big)\,,\quad D_{1234}[15][25][35][45]\,. \qquad (5.17)$$

These terms will multiply some extra tree-level contractions and then one has to sum over the three inequivalent permutations of 1234 and over the choice of other quadruples.

The last term in (5.17) has no orientation, so we expect it to cancel once D_{1243} and D_{1324} are included. The two other terms in (5.17) are the ingredients that make up the minimal four-point function (4.4) and five-point function (5.4) (note the relations in (4.2) and (5.2)). As mentioned, in the case of the general four-point function it turns out that the combinatorics are such that these terms exactly factorize as in (1.1).

For the general five-point function we do not have such a general proof, but for all the cases listed above, and for the those we computed in appendix A, all the graphs can be reorganized in a simple manner. They are all given as the sum of the minimal five-point function (5.4) and the five possible minimal four-point functions (4.4), each multiplied by certain tree-level contractions.

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It would be interesting to check if this decomposition indeed holds beyond the examples computed in appendix A and whether a similar structure persists to higher-loop order.

6 Six-point functions

Lastly we evaluate one six-point function

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\mathcal{O}_2^{u_6}\rangle_{\text{1-loop}} \tag{6.1}$$

All the disc amplitudes appearing in (3.25) are equivalent, up to permutations of the indices. For example the amplitude multiplying D_{1234} is

$$\begin{aligned}\langle \mathcal{O}_1^{u_1}\mathcal{O}_1^{u_2}\mathcal{O}_1^{u_3}\mathcal{O}_1^{u_4}|\mathcal{O}_2^{u_5}\mathcal{O}_2^{u_6}\rangle_{\text{tree, disc}} = 4\Big(&([15][36]+[16][35])([25][46]+[26][45])\\ &+[12][56]([36][45]+[35][46])+[14][56]([25][36]+[26][35])\\ &+[15][56]([26][34]+[23][46])+[16][56]([25][34]+[23][45])\Big).\end{aligned} \tag{6.2}$$

Note that now there are also two possible disconnected graphs $[56]^2([12][34]+[14][23])$, which we have not included.

As usual it is simpler to subtract all graphs, including non-planar ones, and get the result as minus the non-planar graphs

$$\begin{aligned}-64D_{1234}\Big([13][56]\big([25][46]+[26][45]\big)+[24][56]\big([15][36]+[16][35]\big)\\ [15][35][26][46]+[16][36][25][45]\Big).\end{aligned} \tag{6.3}$$

Summing over different permutations of these Wick-contractions with the relevant D_{ijlm} factors gives the one-loop six-point amplitude.

For simple six-point function we expect a similar decomposition as we have found in the sample five-point functions we have studied. Now the four boundary insertions on the disc can be contracted directly, giving terms like the minimal four-point function (4.4), through one of the internal vertices, giving terms proportional to the minimal five-point function (5.4) or through both internal points giving the ingredients of the minimal six-point function (6.1). We do not know whether such considerations hold for arbitrary six-point functions, or for that matter for larger n-point functions.

7 Extremal and next-to-extremal n-point functions

Finally we consider a special class of n-point functions

$$\left\langle \mathcal{O}_{k_1}^{u_1}\mathcal{O}_{k_2}^{u_2}\dots\mathcal{O}_{k_{n-1}}^{u_{n-1}}\mathcal{O}_{k}^{u_n}\right\rangle \qquad \text{with} \quad k=\sum_{i=1}^{n-1}k_i - m\,. \tag{7.1}$$

For $m=0$ these are known as the extremal and for $m=2$ as the next-to-extremal n-point functions, which do not receive quantum corrections at one-loop order.[3] It is instructive

[3]The case $m=1$ vanishes trivially.

to rederive this fact from (3.25). We have

$$\left\langle \mathcal{O}^{u_1}_{k_1}\mathcal{O}^{u_2}_{k_2}\ldots\mathcal{O}^{u_{n-1}}_{k_{n-1}}\mathcal{O}^{u_n}_{k}\right\rangle_{\text{1-loop}} = \tag{7.2}$$

$$\sum_{ijl} D_{ijln}\, k_i k_j k_l k \left\langle \mathcal{O}^{u_i}_{k_i-1}\mathcal{O}^{u_j}_{k_j-1}\mathcal{O}^{u_l}_{k_l-1}\mathcal{O}^{u_n}_{k-1}\middle| \prod_{p\neq i,j,l} \mathcal{O}^{u_p}_{k_p}\right\rangle_{\text{tree, disc}}$$

$$+\sum_{ijlm} D_{ijlm}\, k_i k_j k_l k_m \left\langle \mathcal{O}^{u_i}_{k_i-1}\mathcal{O}^{u_j}_{k_j-1}\mathcal{O}^{u_m}_{k_l-1}\mathcal{O}^{u_m}_{k_m-1}\middle|\mathcal{O}^{u_n}_{k} \prod_{p\neq i,j,l,m} \mathcal{O}^{u_p}_{k_p}\right\rangle_{\text{tree, disc}},$$

where the sums run over all orders of ijl and $ijlm$ respectively. Due to the special structure of (7.1) with $\mathcal{O}^{u_n}_k$ having the largest dimension, all the other $(n-1)$ operators $\mathcal{O}^{u_i}_{k_i}$ of lower dimension have to contract with $\mathcal{O}^{u_n}_k$. Hence the tree-level disc amplitudes in the second line of (7.2) vanish, as these contractions leave four $(m=0)$ or two $(m=2)$ legs of $\mathcal{O}^{u_n}_k$ uncontracted. The same argument also kills the first sum in the extremal case $(m=0)$, which leaves two legs of $\mathcal{O}^{u_n}_k$ uncontracted.

In order to see the vanishing of the first sum in (7.2) in the next-to-extremal case $(m=2)$ we note that the relevant tree-level disc amplitude

$$\left\langle \mathcal{O}^{u_i}_{k_i-1}\mathcal{O}^{u_j}_{k_j-1}\mathcal{O}^{u_l}_{k_l-1}\mathcal{O}^{u_n}_{k-1}\middle| \prod_{p\neq i,j,l} \mathcal{O}^{u_p}_{k_p}\right\rangle_{\text{tree, disc}} = [in]^{k_i-1}[jn]^{k_j-1}[kn]^{k_l-1}\prod_{p\neq i,j,l} k_p [np]^{k_p} \tag{7.3}$$

is independent of the ordering of the labels ijl. Hence

$$\left\langle \mathcal{O}^{u_1}_{k_1}\mathcal{O}^{u_2}_{k_2}\ldots\mathcal{O}^{u_{n-1}}_{k_{n-1}}\mathcal{O}^{u_n}_{\sum_i k_i-2}\right\rangle_{\text{1-loop}} = \tag{7.4}$$

$$\sum_{(ijl)} (D_{nijl}+D_{njil}+D_{nilj})\, k_i k_j k_l k\, [in]^{k_i-1}[jn]^{k_j-1}[kn]^{k_l-1}\prod_{p\neq i,j,l} k_p[np]^{k_p} = 0\,,$$

where the sum is now over the ordered triples (ijl) and the vanishing is due to the relation (3.26) for the D's.

We therefore verified by our formalism the vanishing of the one-loop corrections to the extremal and next-to-extremal correlators.

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A A selection of one-loop five-point functions

We collect here some more explicit results for the one-loop planar five-point functions of chiral primary operators. We only write down the one-loop part. The tree level is found by simple Wick-contractions. They are all expressed in terms of the one-loop four point function and five-point function of operators of dimension two (4.4), (5.4), multiplied by some tree level contractions. We do not write down explicitly the space-time point of each operator, the position x_i is always matched with the index of u_i

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_6^{u_5}\rangle_{\text{1-loop}} = 0 \tag{A.1}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_5^{u_5}\rangle &= 15[45]^3\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle + \frac{45}{2}[35]^2[45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{45}{2}[25]^2[45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + \frac{45}{2}[15]^2[45]\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.2}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_4^{u_4}\mathcal{O}_4^{u_5}\rangle &= 4[45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + 16[34][35][45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+16[24][25][45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + 16[14][15][45]\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.3}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_3^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_4^{u_5}\rangle &= \frac{9}{2}[35][45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+9[34][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle + 9[34][35]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{9}{2}[25](2[34][25] + [35][24] + [23][45])\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{9}{2}[15](2[34][15] + [35][14] + [13][45])\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.4}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_3^{u_2}\mathcal{O}_3^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_3^{u_5}\rangle &= \frac{81}{16}([23][45] + [24][35] + [25][34])\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{81}{8}([25,34|1] + [23,45|1] + [24,35|1])\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle\,. \end{aligned} \tag{A.5}$$

$$\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_7^{u_5}\rangle = 0 \tag{A.6}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_4^{u_4}\mathcal{O}_6^{u_5}\rangle &= 24[45]^4\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle + 48[34]^2[45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+48[24]^2[45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + 48[14]^2[45]^2\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.7}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_3^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_6^{u_5}\rangle &= \frac{81}{2}[34][45]^3\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle + \frac{81}{2}[35]^3[45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+54[25]^2[35][45]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + 54[15]^2[35][45]\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.8}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_5^{u_4}\mathcal{O}_5^{u_5}\rangle &= \frac{25}{4}[45]^3\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{75}{2}[35][35][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle + \frac{75}{2}[25][25][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{75}{2}[15][15][45]^2\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.9}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_3^{u_3}\mathcal{O}_4^{u_4}\mathcal{O}_5^{u_5}\rangle &= \frac{15}{2}[35][45]^2\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+30[34][45]^3\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle + 30[35]^2[45][34]\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{15}{2}[25][45](6[25][34] + 4[24][35] + 2[23][45])\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{15}{2}[15][45](6[15][34] + 4[14][35] + 2[13][45])\langle \mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \end{aligned} \tag{A.10}$$

$$\begin{aligned}\langle \mathcal{O}_2^{u_1}\mathcal{O}_3^{u_2}\mathcal{O}_3^{u_3}\mathcal{O}_3^{u_4}\mathcal{O}_5^{u_5}\rangle &= \frac{135}{16}\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \\ &+\frac{135}{8}[45]^2(2[23][45] + [24][35] + [25][34])\langle \mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_5}\rangle \end{aligned}$$

$$+\frac{135}{8}[35]^2(2[24][35]+[23][45]+[25][34])\langle\mathcal{O}_2^{u_1}\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle$$
$$+\frac{135}{8}[25]^2(2[25][34]+[24][35]+[23][45])\langle\mathcal{O}_2^{u_1}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle$$
$$+\frac{135}{8}[15](3[15][25][34]+[12][35][45]+[14][25][35]+[13][25][45])\langle\mathcal{O}_2^{u_2}\mathcal{O}_2^{u_3}\mathcal{O}_2^{u_4}\mathcal{O}_2^{u_5}\rangle \tag{A.11}$$

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